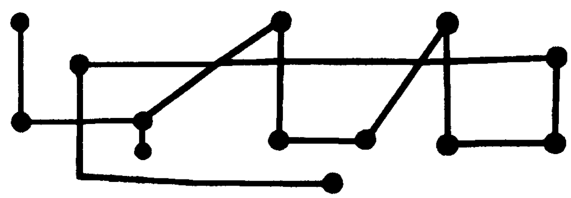
GRAPH-THEORETICAL INSIGHTS INTO THE HORIZONTAL-VERTICAL DUALITY OF MUSIC



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INTRODUCTION

This paper engages the duality between harmony and melody, representing the vertical and horizontal dimensions of music, respectively. Our investigation endeavors to establish a method for encoding the rules governing both harmonic and melodic dimensions within a composition using graph theory. This formalism will provide composers with the means to structure the harmonic and melodic dimensions of a composition according to distinct strategies, as we will provide a method for systematically assessing the compatibility between the melodic and harmonic syntax upon which a composition is based.

In numerous instances of post-tonal music where pitch is systematically considered, the conventional duality between harmony and melody dissolves. Rather than adhering to distinct syntaxes, harmony in such compositions is frequently derived from melodic structures in various ways. One method involves collapsing a linear series of pitches or pitch classes onto a single point in time, leading to the formation of simultaneities, as discussed in [12, pages 55–60].

Another method is afforded by composing with pitch-class arrays. In this case, potential simultaneities are determined by the columns of an array.[1] For instance, the first column of the array in Example 1 facilitates a vertical arrangement of four distinct pitch-class sets: $\{0,3,7\}$, $\{0,6,7\}$, $\{1,3,7\}$, and $\{1,6,7\}$. While composing with such arrays grants the composer with considerable freedom, it remains the case that harmonic decisions are made subsequent to the melodic decisions that gave rise to the rows (and therefore columns) of the array.

01	42
36	19
7	867

EXAMPLE 1

The approach presented in this paper diverges from the aforementioned methods by conceptualizing the horizontal and vertical dimensions of a composition as governed by distinct syntaxes, each defined using mathematical graph theory. This methodology offers several advantages: (1) it allows us to conceive the harmonic and melodic dimensions of a piece of music independently, and (2) it facilitates a systematic approach to assessing the compatibility between the two dimensions. Thus, rather than emphasizing one dimension over the other and deriving the less emphasized dimension from the former, our approach grants the flexibility to tailor both dimensions according to unique compositional strategies. Subsequently, we can assess their compatibility through mathematical methods.

To accomplish our objective, we will structure our paper as follows:

1. In Section 1, we provide the requisite concepts from graph theory.

2. Subsequently, Section 2 introduces a precise method for encoding melodic and harmonic syntaxes using graph theory.

3. Building upon Morris' work [8] regarding voice leading between pitch-class sets, Section 3 demonstrates how to encode such voice leadings within the proposed graph-theoretic framework.

4. In Section 4, we introduce the concept of a *tonal context*, which is a pair consisting of a melodic and a harmonic syntax. Following this definition, we provide methods for assessing the extent to which the melodic and harmonic syntaxes of a tonal context are compatible with each other—specifically, whether moves in the melodic syntax can coincide with moves in the harmonic syntax, and vice versa.

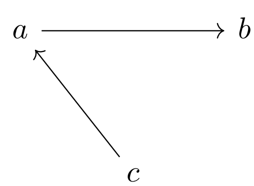
5. Section 5 concludes the paper, offering suggestions for future research.

This approach will equip us with the tools to systematically analyze the compatibility between the horizontal and vertical dimensions of music.

1. DIRECTED GRAPHS IN MUSIC THEORY

In this section we provide the reader with the necessary background information from graph theory. We present the rigorous mathematical framework, as the explicitness that it affords will ultimately enable a more systematic method for conceiving of the melodic and harmonic syntaxes that we will present in Section 2.

Informally, a directed graph (digraph) is a mathematical structure consisting of a set of vertices and a set of directed arrows that connect the vertices (Example 2). Such structures have been extensively employed in the music-theory literature, with sources like [2, 3, 6, 7, 8] offering a rich background. Additionally, more recent works, such as [13], utilize graph theory in novel ways.

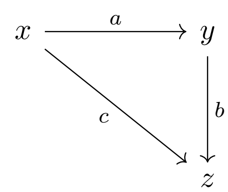


EXAMPLE 2

For our purposes, we define digraphs explicitly using the following set-theoretic definition.

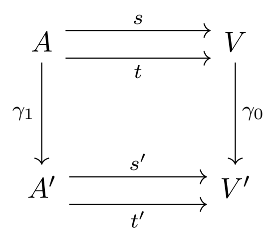
Definition 1 (Directed graph). A *directed graph*, or *digraph*, is a quadruple (V, A, s, t) where V and A are sets, and $s : A \to V$ and $t : A \to V$ are set functions where s gives the source vertex of an arrow and t gives its target vertex.

An example of a digraph is the following. Let $V = \{x, y, z\}$ be a vertex set and $A = \{a, b, c\}$ an arrow set. Define the source map $s : V \to A$ such that $s(a) = s(c) = x$ and $s(b) = y$. Next, define the target map $t : V \to A$ such that $t(a) = y$ and $t(b) = t(c) = z$. Then the digraph (V, A, s, t) can be visualized as in Example 3.



EXAMPLE 3

The definition of a digraph gives rise to a mathematical category[2] $Digr$ of digraphs. A morphism $\Gamma : (V, A, s, t) \to (V', A', s', t')$ in $Digr$ is given by a pair of morphisms γ_0, γ_1 on the vertex and arrow sets, respectively, such that the following diagram commutes.



The essence of such a morphism lies in preserving the graphical structure by maintaining the arrow relation between vertices.

An alternative method to define a digraph is to combine the source and target maps into a single function $d : A \to V \times V$, where $d(a) = (u, v)$ indicates that a is an arrow from u to v. This approach is equivalent to the previous method, as the fact that $d(a) = (u, v)$ is equivalent to the fact that $s(a) = u$ and $t(a) = v$. Henceforth, we will define digraphs using the single morphism approach rather than the source and target method. This distinction will become important later, as the single morphism approach allows us to define a digraph simply as a single set.

Before proceeding, it is crucial to highlight a potential scenario. There may be instances where we need duplicate elements of the vertex and/or arrow set need to occur in multiple locations in a digraph. Consider the digraphs in Example 4. The vertex sets of both digraphs contain elements from $\mathbb{Z}_{12}$, used to denote pitch classes, whereas the arrow sets contain elements from the group T/I of transpositions and inversions. Now consider the digraph in Example 4a, where two instances of pitch-class 1 are present. If we were to define its digraph map as $d(T_3) = (1, 4)$ and $d(T_9) = (4, 1)$, it would result in the digraph shown in Example 4b. Notably, the latter digraph exhibits a different graphical structure compared to Example 4a—the former being linear, while the latter is cyclical.

(a) (b)

EXAMPLE 4: TWO DIGRAPHS WITH THE SAME VERTICES AND ARROWS, BUT DIFFERENT GRAPHICAL STRUCTURES.

To enable definition of digraphs such as those in Example 4a, we must implement a method for duplicating elements in the vertex (and arrow) sets. Fortunately, category theory provides a solution through a construction called a *coproduct*. For any sets A and B, their *coproduct* $A \sqcup B$ yields their disjoint union. For example, for the set A, its coproduct with itself $A \sqcup A$ is the set consisting of two copies of each $a \in A$.[3] Consequently, when aiming to construct a digraph with vertex set V and arrow set A, we typically define our actual vertex and arrow sets as subsets of the infinite-fold coproducts of V with itself and A with itself, notated respectively as

$$V' \subset \coprod^{\infty} V$$

and

$$A' \subset \coprod^{\infty} A.$$

Thus, the digraphs in 4a and 4b display distinct vertex sets. The former, $V_a = \{1_1, 1_2, 4\}$, contains two instances of $1 \in \mathbb{Z}_{12}$, indexed by different subscripts, while the latter, $V_b = \{1, 4\}$, contains a single instance of $1 \in \mathbb{Z}_{12}$. Both digraphs share the same arrow set, $A = \{T_3, T_9\}$.

Before proceeding, we demonstrate how to encode digraphs simply as sets, rather than quadruples (V, A, s, t) or triples (V, A, d). To achieve, we define the *limit* of a digraph function $d : A \to V \times V$ as follows:

$$\lim d = \{(a, (u, v)) \mid a \in A \text{ and } u, v \in V \text{ such that } d(a) = (u, v)\}.$$

This set consists of pairs where the first coordinate is an arrow a and the second coordinate is a pair of vertices (u, v) such that a connects u to v.

2. ENCODING MELODIC AND HARMONIC SYNTAX USING DIGRAPHS

In this section, we illustrate how to encode melodic and harmonic syntaxes as digraphs. The concept of syntax revolves around a set of rules used to generate valid sequences. Typically, one specifies an alphabet A, and the syntax of a language comprises rules for arranging the elements of A into valid sequences. We can represent the syntax of a language using digraphs, where an arrow between vertices $x \in A$ and $y \in A$ signifies the ability for y to follow x in a sequence.

Consequently, a valid string of elements is conceptualized as any path in a digraph G, and the set of all valid sequences corresponds to the set of paths in G.

In the case of melodic syntax, we can envision a digraph with a vertex set consisting of pitches or pitch classes to represent such a syntax. Similarly, a harmonic syntax can be portrayed by a digraph with a vertex set consisting of pitch or pitch-class sets. To derive such vertex sets, we first define a set Z of pitches or pitch classes, and then take its power set $\mathcal{P}(Z)$ to derive the set of pitch or pitch-class subsets of Z. We can then define a *melodic syntax* as a digraph with a vertex set as any subset of the infinite-fold coproduct of Z with itself, and a *harmonic syntax* as a digraph with a vertex set as any subset of the infinite-fold coproduct of $\mathcal{P}(Z)$ with itself.

3. ENCODING VOICE LEADINGS BETWEEN PITCH-CLASS SETS

Now that we have established the graph-theoretic foundations for conceiving harmonic and melodic syntaxes, we can align some of the concepts involving voice leading between pitch-class sets defined by Morris in [8] with the concepts presented in this paper. Morris [8, page 178] defines the *total voice leading* between two pitch-class sets A and B as follows:

> Given two pitch-class sets A and B, the *total voice leading* from A to B includes any and all moves from any pitch classes of A to any pitch classes of B—that is, all the ways one can associate the pitch classes of A with those of B in as many voices as necessary or desired. Each voice will be a path from a pitch class or pitch-class subset of A to B.

From a mathematical standpoint, the definition is vague. To make the definition explicit, we present a formal definition of total voice leading in terms of digraphs.

A voice leading from A to B can be modeled as a digraph, constructed as follows. Morris allows for any move from any pitch classes of A to any pitch classes of B, in as many voices as desired. This implies that a single voice, for instance, can play multiple pitch classes from A or B, or that multiple voices can play the same pitch class. To construct the vertex set, we start with pitch-class subsets $A, B \subset \mathbb{Z}_{12}$. Since a single voice can play multiple pitch classes from A or B, we need subsets of A and B. Also, since multiple voices can play the same pitch class or pitch-class subset from A or B, we need to take coproducts that duplicate such pitch classes and pitch-class subsets. Therefore, to encode a voice leading between A and B as a digraph, we break up the vertex set into two components, namely

$$V_A \subset \coprod^{\infty} \mathcal{P}(A)$$

and

$$V_B \subset \coprod^{\infty} \mathcal{P}(B).$$

Now let E be an arrow set. Then a voice leading from A to B is encoded as a digraph via a set function of the following form:

$$\Gamma : E \rightarrow V_A \times V_B.$$

The notation reflects that the mapping Γ assigns an arrow $e \in E$ to a pair $(a, b) \in V_A \times V_B$ where a is a pitch class or pitch-class subset of A and b is a pitch class or pitch-class subset of B, following Morris' definition of a voice leading from A to B. The *total* voice leading from A to B therefore is the set of all digraphs of such a form.

To examine concrete examples of voice leadings, refer to Example 5, sourced from [8]. Let's illustrate how we would explicitly encode the voice leading in 5a. In 5a, the voice leading begins with three voices and transitions to two voices. The top voice plays two pitch classes from A, while the bottom two voices play the same pitch class from A. Subsequently, the top and bottom voices play one pitch class from B, while the middle voice drops out. This results in the vertex sets

$$V_A = \{\{\text{B}\}_2, \{\text{B}\}_3, \{2, \text{A}\}\}$$

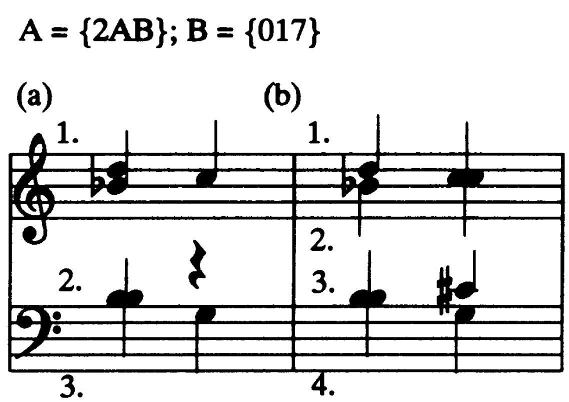
and

$$V_B = \{\{0\}, \{7\}, \{\}\}.$$

Let $E = \{e_1, e_2, e_3\}$ be the arrow set. We then define the map $\Gamma : E \rightarrow V_A \times V_B$, defined on the elements of E as follows:

$$\Gamma(e_1) = (\{2, \text{A}\}, \{0\}),$$
$$\Gamma(e_2) = (\{\text{B}\}_2, \{\}),$$
$$\Gamma(e_3) = (\{\text{B}\}_3, \{7\}).$$

Hence, the voice leading from Example 5a is encoded by the digraph given by Γ.

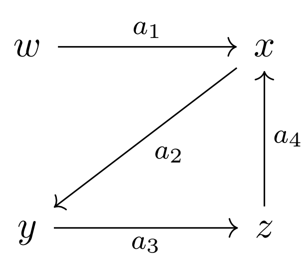


EXAMPLE 5: TWO ILLUSTRATIONS OF VOICE LEADINGS FROM PITCH-CLASS SET A TO B, TAKEN FROM MORRIS [8].

4. ASSESSING THE COMPATIBILITY BETWEEN A HARMONIC AND A MELODIC SYNTAX

This section culminates the preceding discussions by offering a method to evaluate the compatibility of a harmonic syntax with a melodic syntax, which is essential for composing using this dual approach: one for the piece's harmonic dimension and the other for its melodic dimension. This inquiry is pivotal because the two dimensions could potentially be entirely incompatible, with melodic moves in the melodic syntax failing to align with the harmonic progressions facilitated by the harmonic syntax, or vice versa. Therefore, we define a *tonal context* as a pair (M, H), where M and H represent, respectively, the harmonic and melodic syntaxes that govern a musical piece or section thereof. For simplicity, we refer to the vertices in M as 'pitch classes' and those in H as 'pitch-class sets', although they may also be simply pitches or pitch sets in full generality. Henceforth, we will use the variables M and H to denote arbitrary melodic and harmonic syntaxes.

Before proceeding, we introduce the concept of a *path* in a digraph. Intuitively, a path is a sequence of arrows $a_1, \dots, a_k$ in a digraph such that the target vertex of each a_i is the source vertex of a_{i+1}. For instance, consider the following digraph.



Here, the sequence $a_1 a_2 a_3 a_4$ forms a path, whereas $a_2 a_1$ does not.

Alternatively, we can define a path as a sequence of vertices. The example above then becomes the vertex sequence $wxyzx$. It is important to note that encoding a path as a series of arrows contains one less arrow than the number of vertices when encoded as a series of vertices. While conceptualizing paths as sequences of vertices is perhaps more intuitive, the set-theoretic encoding of digraphs—wherein paths are defined by taking the set-theoretic limit of the digraph morphism $d : A \to V \times V$—necessitates the arrow approach for rigorous definition. Nonetheless, we determine the *length* of a path by counting its vertices.

Formally encoding a path of length K involves defining the set K of natural numbers from 1 to $K - 1$. For a digraph represented by the morphism $d : A \to Z \times Z$, we encode it as the set $G = \lim\ d$, consisting of pairs $\big(a, (x, y)\big)$ such that $d(a) = (x, y)$, as defined in Section 1. A *path* in G is then defined as an injective function $p : K \to G$, satisfying the condition that for every $i \in K$, if $p(i) = \big(a, (x, y)\big)$, then $p(i + 1) = \big(b, (y, z)\big)$. This condition ensures that the target of the ith arrow corresponds to the source of the $(i + 1)$th arrow.

Although mathematical clarity dictates the encoding of paths as sequences of arrows rather than vertices, we can devise a canonical method to translate sequences of arrows into sequences of vertices. For a digraph given by the function $d : A \to V \times V$, its graphical structure and collection of vertices can be encoded such that for each vertex $v \in V$, there exists an arrow $\mathrm{id}_v \in A$ satisfying $d(\mathrm{id}_v) = (v, v)$. Essentially, these arrows serve to identify the vertices, as each points from a vertex to itself.

Thus, for a path of arrows $p : K \to G$, where $G = \lim\ d$, we can conceive that each arrow $p(i) = (a, (x, y))$ represents the source vertex x of a, and the last arrow $p(K) = (\mathrm{id}_v, (v, v))$ is an arrow from the last vertex in the path to itself. For instance, consider the following path:

$$x \overset{a}{\to} y \overset{b}{\to} z$$

It is encoded as the sequence $(a, (x, y)), (b, (y, z)), (\mathrm{id}_z, (z, z))$, which corresponds bijectively with the sequence of vertices x, y, z. This translation is necessary as we will later perform technical operations on sets, necessitating the explicit encoding of paths as sets.

Now we work toward explicating the conditions under which a melodic syntax M is compliant with a harmonic syntax H. We will first define the conditions under which a path in a melodic syntax M is compliant with a single chord transition $h \overset{a}{\to} h'$ in a harmonic syntax H. This will allow us to then define whether a chord transition $h \overset{a}{\to} h'$ is 'covered' by a collection of paths in M, meaning that the entire chord h sounds and then the entire chord h'. These results are then generalized to arbitrary chord progressions (paths) afforded by H.

Before proceeding, we present a notational convention. For a path $p : K \to G$, denote by p_i the ith vertex in the path, where p_1 is the first vertex. We will also write $p_\#$ to denote the last vertex in p, for p any arbitrary path.

Definition 2 (Compliant). Let $h \overset{a}{\to} h'$ be an arrow in a harmonic syntax H and p a path in a melodic syntax M. We say that p and a are *compliant*, or that p *complies with* a, if and only if there exists an $m \in h$ such that $p_1 = m$ and there exists an $m' \in h'$ such that $p_\# = m'$.

More informally, a path of pitch classes in M is compliant with a series of two pitch-class sets h, h' if the path starts with a pitch class in h and ends with a pitch class in h'.

We are particularly interested in cases where we have a set of paths P in M each of which is compliant with an arrow $h \overset{a}{\to} h'$ in H, and such that the set of starting pitch classes is equal to h and the set of ending pitch classes is equal to h'. We refer to such a set P as a *cover* of a. More formally, the definition is as follows.

Definition 3 (Cover-1). A *cover* of an arrow $h \overset{a}{\to} h'$ in H is a set P of paths in M such that $\{p_1 \mid p \in P\} = h$ and $\{p_\# \mid p \in P\} = h'$.

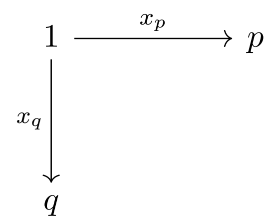
To see an example, let $h = \{0, 4, 5\}$ and $h' = \{1, 2, 3\}$ be pitch-class sets such that there is an arrow $h \overset{a}{\to} h'$ in H. Let $P = \{p^1, p^2, p^3\}$ be a set of paths in M, defined as follows:

$$p^1 \coloneqq 0 \to 7 \to 2,$$
$$p^2 \coloneqq 4 \to \mathrm{A} \to 9 \to 3,$$
$$p^3 \coloneqq 5 \to 1.$$

Then P covers a.

We now aim to generalize the above definitions so that they apply to arbitrary paths of pitch-class sets in H. To achieve this, we proceed as follows.

Consider paths p and q, where $p_\# = q_1 = x$, signifying that the path p concludes with the vertex x from which q commences. In such instances, we can 'glue' p and q at their common point x using a set-theoretic *colimit*. Specifically, we define two morphisms $x_p : 1 \to p$ and $x_q : 1 \to q$ in the category **Set** of sets, where $1 = \{*\}$ represents the singleton set. These two morphisms from a common domain are represented by the following diagram[4] $\mathcal{D}$ in **Set**.



The set-theoretic *colimit* of $\mathcal{D}$, denoted by $p +_1 q$, is the set-theoretic union of p and q modulo the equivalence relation $\sim$ defined by $p_i \sim q_j$ if and only if $x_p(*) = x_q(*)$. In other words, the colimit $p +_1 q$ is the union of p and q with the terminal vertex of p identified with the initial vertex of q. Therefore $p +_1 q$ is the path that results from gluing the terminal vertex of p to the initial vertex of q, effectively merging p and q into a single path.

For a sequence of paths $p^1, \dots, p^n$, if every path satisfies $p^i_\# = p^{i+1}_1$, then we can glue them together via the colimit construction. We denote such a colimit construction as

$$\sum_{i=1}^{n} p^i.$$

For instance, consider the following paths:

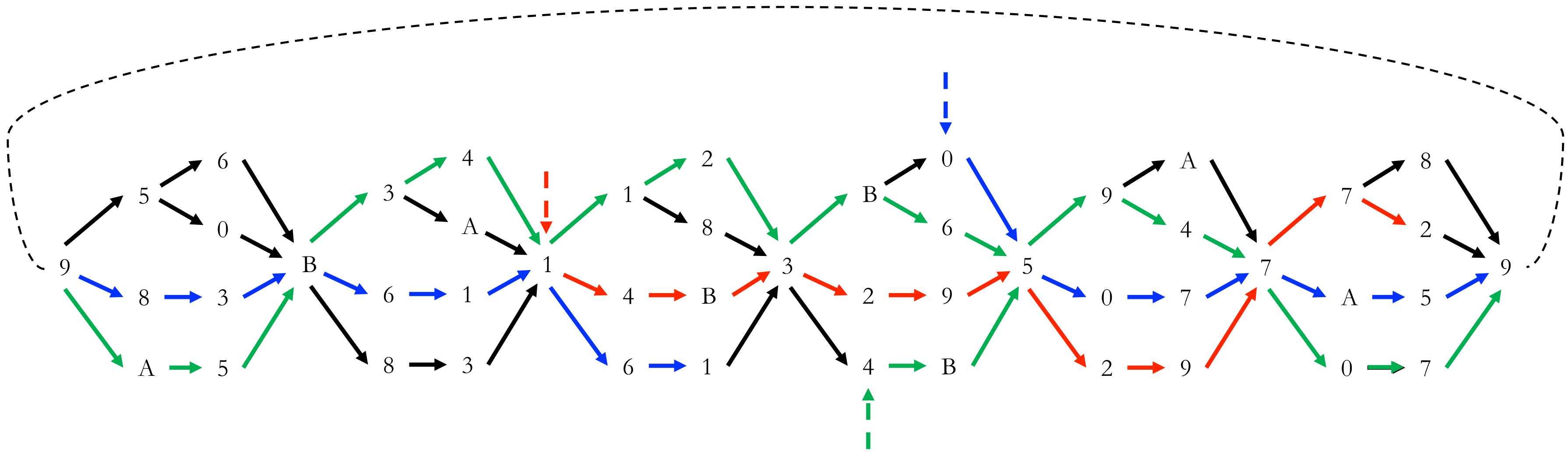
$$\begin{aligned} p^1 &:= a \to b \to c, \\ p^2 &:= c \to d, \\ p^3 &:= d \to e \to f. \end{aligned}$$

The colimit obtained by successive gluings is defined as the path $p = \sum_{i=1}^{3} p^i$, which corresponds to $a \to b \to c \to d \to e \to f$.

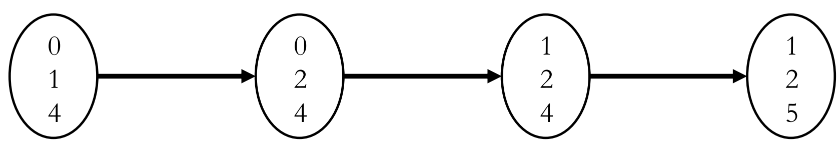
Conversely, given a path p, we can derive a series of subpaths $p^1, \dots, p^n$ such that their gluing equals p:

$$\sum_{i=1}^{n} p^i = p.$$

A collection of subpaths satisfying this condition is termed a *partition* of p, and each subpath is termed a *component* of the partition. For instance, the set of paths $\{p^1, p^2, p^3\}$ from above constitutes a partition of the path p derived from their gluing, where p^1, p^2, and p^3 are the components of the partition.



(a) A MELODIC SYNTAX M. THE PATHS p, q, AND r ARE DEPICTED BY THE BLUE, RED, AND GREEN ARROWS, RESPECTIVELY. THE DOTTED BLACK LINE CONNECTING THE FIRST 9 WITH THE LAST INDICATES THEIR EQUIVALENCE, ESTABLISHING THE CYCLIC NATURE OF THE ENTIRE GRAPH..



(b) A PATH π OF PITCH-CLASS SETS IN A HARMONIC SYNTAX H.

EXAMPLE 6

Now, our current objective is to leverage these concepts of ‘gluing’ and ‘partition’ to generalize the notion of a *covering* introduced earlier. Whereas our previous discussion was confined to a single transition of pitch-class sets $h \xrightarrow{a} h'$ in a harmonic syntax H, we now aim to extend the concept of a covering to arbitrary paths in H.

Definition 4 (Cover-2). Let π be a path in H of length n. A *cover* of π consists of a set P of paths in M, equipped with partitions $\Phi(p)$ of size $n-1$ for each $p \in P$ such that the following condition is satisfied:

- Define the set

$$\Phi_j^i \coloneqq \left\{ \left(\Phi(p)^i \right)_j \mid p \in P \right\},$$

consisting of the jth vertex of each ith component of the partitioned paths in P. Then, for π_i in π, with $1 \leq i < n$, we require that $\Phi_1^i = \pi_i$. Furthermore, for the last harmony π_n in π, we require $\Phi_{\#}^n = \pi_n$.

Hence a cover of a harmonic progression consists of a set of melodic voices capable of manifesting the progression’s harmonies through their vertical alignment. Of coures, it’s possible that for such a path π in H, a cover derived from M may not exist. In such instances, M and H are deemed incompatible syntaxes, as certain harmonic progressions in H cannot be realized by the melodic potentialities offered by M. Hence, we provide a definition.

Definition 5 (Compatible). For a tonal context (M, H), we say that M and H are *compatible* if and only if for every path $\pi \in H$, M admits a cover of π.

A potential area for further research involves exploring different types of compatibilities between melodic and harmonic syntaxes. This could entail developing more nuanced classifications of the types of compatibilities that may arise. Additionally, examining degrees of compatibility, such as determining the number of paths in H that M can cover, presents another avenue for investigation.

We conclude this section with a musical example of a cover (in the sense of **Definition 4**). This rather trivial example showcases how the harmonic and melodic dimensions of a composition can be structured indepedently, each following distinct strategies, before converging to shape the music’s surface. Example 6a provides a melodic syntax M, while Example 6b presents a path π of pitch-class sets, presumably situated within a larger ambient harmonic syntax H. A cover of π can be derived from paths p, q, and r in M, with their respective partitions represented by the columns in the following array.

p:	$\underline{0}5$	$\underline{0}77A5983B6$	$\underline{1}16\underline{1}$
q:	$\underline{1}4B3$	$\underline{2}95$	$\underline{2}977\underline{2}$
r:	$\underline{4}B59$	$\underline{4}7079A5B3$	$\underline{4}1123B6\underline{5}$

It is evident that the melodic and harmonic dimensions follow distinct compositional strategies. The harmonic strategy features a series of pitch-class sets, each derived from the preceding one by shifting one pitch class up or down a semitone. Notably, while the initial and final pitch-class sets share the same set class (014), the intermediate pitch-class sets represent distinct set classes—specifically, (024) and (013). Consequently, the harmonic strategy can be characterized as a process in which a harmony undergoes iterative 'deformation', where its intervallic structure is not preserved, ultimately resulting in a 'transformed' version of its initial self (in this case, a transposition, which preserves the interval structure of the initial chord).

On the other hand, the melodic dimension is characterized by featuring many instances of pitch-class interval 7 in paths p and q, and pitch-class interval 1 in path r. The melodic and harmonic strategies converge to shape the surface of the music, characterized by static harmonies that are interrupted by bursts of melodic motion (Example 7).

5. CONCLUSION

This paper aims to provide a valuable perspective on the interplay between harmony and melody in music, targeting composers and theorists engaged in systematic approaches to composition and analysis. A distinguishing feature of our approach lies in the ability to customize the harmonic and melodic syntaxes of a musical composition, addressing a relatively unexplored aspect in post-tonal music theory and composition.

While existing literature in post-tonal music theory often emphasizes the construction of pitch-class series governing the linear progression of pitch classes over time, our approach seeks to reintroduce the vertical dimension of music as amenable to sophisticated compositional strategies.

Arrays, as generalizations of series, enable the simultaneous unfolding of multiple series, granting composers the freedom to construct chords from array columns. This method offers greater flexibility in harmonic construction compared to a single row. However, in many cases, the treatment of harmony remains a secondary consequence of the linear unfolding of series. In contrast, our approach explicitly addresses both harmonic and melodic strategies. The concepts of 'compliance', 'covering', and 'compatibility' introduced in this paper provide composers with a systematic method to identify compatible harmonic progressions with specific pitch and pitch-class sequences. Future research may involve implementing computer programs to search for these compatibilities, enabling composers to incorporate the results into their compositions with greater ease.

Further exploration in this area could delve deeper into the diverse relationships and compatibilities between harmonic and melodic syntaxes. We hope that our work inspires theorists and composers interested in innovative approaches to the melodic and harmonic dimensions of music within the realm of systematic approaches to composition.

EXAMPLE 7: MUSICAL REALIZATION OF A COVERING.

NOTES

1. Seminal early work on array composition can be found in [1], as well as [7, 8].
2. For background information in category theory see [4]. For the use of categories in music theory, refer to sources such as [5, 6, 9].
3. Since sets don't have duplicate elements, a common way to represent such a disjoint union is by indexing duplicate elements with an ordinal. For instance, two instances of $a \in A$ can be encoded as $(a, 1)$, $(a, 2)$, or a_1, a_2.
4. A *diagram* in a category C is a functor $\mathcal{D} : J \to C$, where J is called the *index category* of the diagram. Essentially, a diagram picks out a set of objects and morphisms in C.

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